Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Chapter 2:

Lecture 5: Random Variables-Introduction, Distributions, Density and Mass Functions

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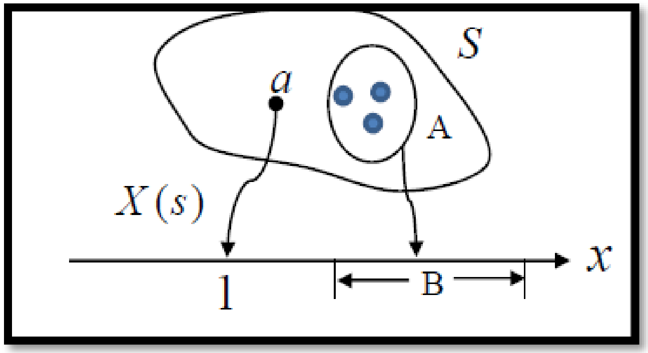
Outlines

1) **The Random Variable Concept, Introduction**
2) **Cumulative Distribution Function (CDF)**
3) **Probability Density and Mass Functions**

The Random Variable Concept, Introduction

- ✓ Variables whose values are due to **chance** are called random variables.
- ✓ A **random variable** (**r.v**) is a real function that maps the set of all experimental **outcomes of a sample space S** into a **set of real numbers.**



- ✓ We shall represent a **random variable** by a **capital letter** (such as X, Y, or W) and any particular **value of the random variable** by a **lower case letter** (such as x, y, or w)
- ✓ Given an experiment defined by a sample space **S** with elements **s**, we **assign** to every **s** a real number **X(s)** according to **some rule** and call **X(s)** a **random variable**
- ✓ There are three types of random variables:

  1) **Discrete Random Variable** (**random** variables have discrete values; the sample space can be discrete, continuous or even mixture of discrete and continuous)

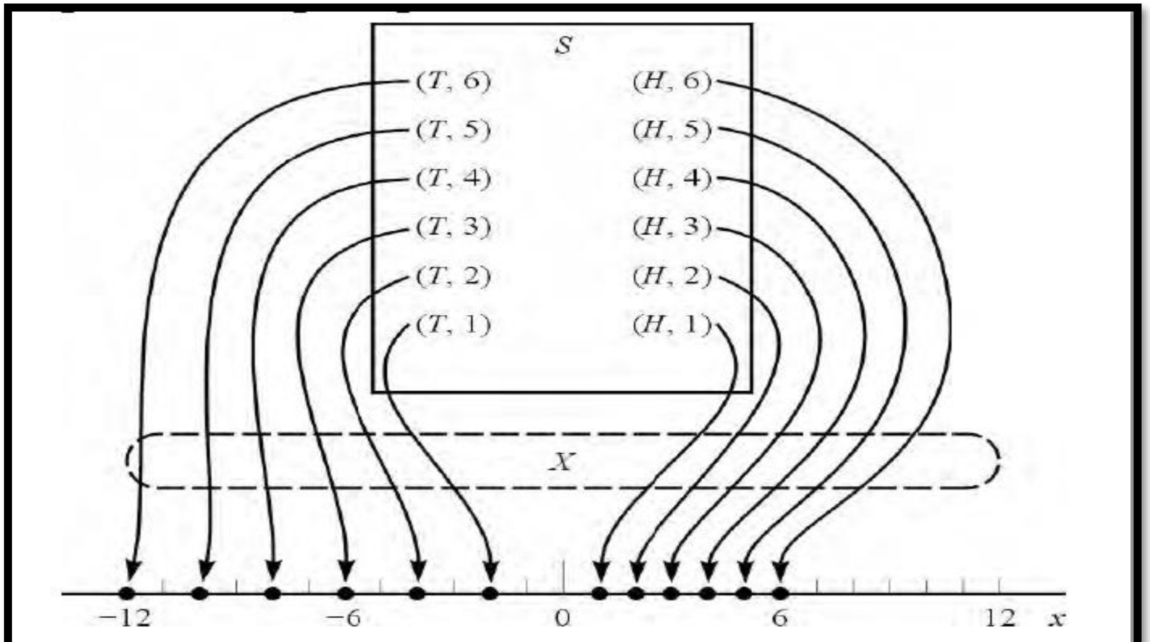
  2) **Continuous Random Variable** (continuous range of values, it cannot be produced from a discrete sample space or a mixed sample space).

  3) **Mixed Random Variables** (less important type of random variables)

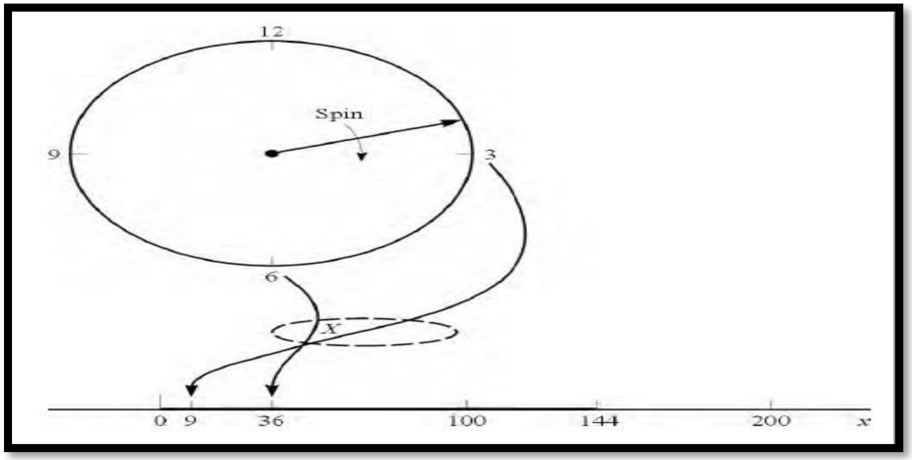
  - **Example**: (**Discrete Random Variable**):
    - An experiment consists of **rolling a die** and **flipping a coin**. The sample space is shown in Fig. below and the random

variable **X** maps the sample space of 12 elements into 12 values of X.

- Function X chosen such that
  - A coin Head (H) outcome corresponds to positive values of X that are equal to the numbers that show up on the die.
  - A coin is Tail (T) outcome corresponds to the negative values of X that are equal in magnitude to twice the number that shows up on the die



- **Example: (Continuous Random Variable)**
  - Figure below illustrates an experiment where the pointer on a wheel of chance is spun. The sample consists of the numbers in the set $\{0 < s \leq 12\}$ and the random variable is defined by the function $X = X(s) = s^2$



✓ **Conditions** for a **Function** to be a **Random Variable**:

- It not be **multivalued**
- The set $\{X \leq x\}$ shall be an **event** for any real number **x**
- $P\{X \leq x\}$ is equal to the sum of the probabilities of all the elementary events corresponding to $\{X \leq x\}$.
- The probabilities of events $\{X = \infty\}$ and $\{X = -\infty\}$ be 0:

$$P\{X = \infty\} = 0 \; P\{X = -\infty\} = 0$$

Probability Distributions

- ✓ **A probability distribution** consists of the values of a random variable and their corresponding probabilities.
- ✓ There are **two kinds** of probability distributions: **discrete** and **continuous.**
- ✓ If **X** is discrete, then the values $P(X = a1), P(X = a2), \ldots$ tell us everything we need to know about **X**.
- ✓ Let **X** be a discrete random variable, and suppose that the possible values that it can assume are given by $x1, x2, x3, \ldots$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$P(X = x_k) = f(x_k) \qquad k = 1, 2, \ldots$$

Probability function, also referred to as **probability mass function (PMF)**, given by

$$P(X = x) = f(x)$$

✓ In general, $f(x)$ is a probability function if

$$1.\ f(x) \geq 0$$
$$2.\ \sum_x f(x) = 1$$

1) The $f(x)$ is a function with nonnegative values
2) The **sum** of the probabilities of a probability distribution must be 1.

- **Example**: When a die is rolled

Value, x	1	2	3	4	5	6
Probability, $P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- **Example**: Construct a discrete probability distribution for the number of heads when three coins are tossed.
- **Solution:**
  - Recall that the sample space for tossing three coins is **TTT**, **TTH, THT, HTT, HHT, HTH, THH**, and **HHH**.
  - The outcomes can be arranged according to the number of heads, as shown.

0 heads TTT
1 head TTH, THT, HTT
2 heads THH, HTH, HHT
3 heads HHH

- Finally, the outcomes and corresponding probabilities can be written in a **table**, as shown.

Outcome, x	0	1	2	3
Probability, $P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

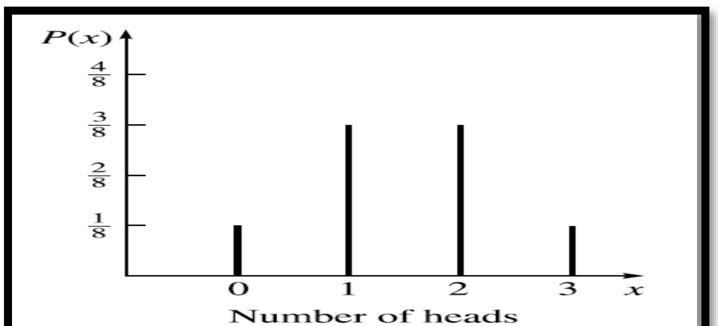
○ Roll two dice, let **Y** be the maximum of their outcomes.

$$P\{Y=1\} = P\{(1,1)\} = 1/36$$
$$P\{Y=2\} = P\{(1,2),(2,1),(2,2)\} = 3/36$$
$$P\{Y=3\} = P\{(1,3),(2,3),(3,1),(3,2),(3,3)\} = 5/36$$
$$P\{Y=4\} = P\{(1,4),(2,4),(3,4),(4,1),(4,2),(4,3),(4,4)\} = 7/36$$
$$P\{Y=5\} = P\{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\} = 9/36$$
$$P\{Y=6\} = P\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} = 11/36$$

- Roll two dice, let **X** be the sum of their outcomes.

$$\begin{aligned}
P\{X=2\} &= P\{(1,1)\} &&= 1/36\\
P\{X=3\} &= P\{(1,2),(2,1)\} &&= 2/36\\
P\{X=4\} &= P\{(1,3),(2,2),(3,1)\} &&= 3/36\\
P\{X=5\} &= P\{(1,4),(2,3),(3,2),(4,1)\} &&= 4/36\\
P\{X=6\} &= P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} &&= 5/36\\
P\{X=7\} &= P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} &&= 6/36\\
P\{X=8\} &= P\{(2,6),(3,5),(4,4),(5,3),(6,2)\} &&= 5/36\\
P\{X=9\} &= P\{(3,6),(4,5),(5,4),(6,3)\} &&= 4/36\\
P\{X=10\} &= P\{(4,6),(5,5),(6,4)\} &&= 3/36\\
P\{X=11\} &= P\{(5,6),(6,5)\} &&= 2/36\\
P\{X=12\} &= P\{(6,6)\} &&= 1/36
\end{aligned}$$

- A discrete probability distribution can also be shown **graphically** by labeling the x axis with the values of the **outcomes** and letting the values on the y axis represent the **probabilities** for the outcomes.



Cumulative probability Distribution Function (CDF)

- ✓ The probability of the event $\{X \leq x\}$ must depend on **x**. Denote

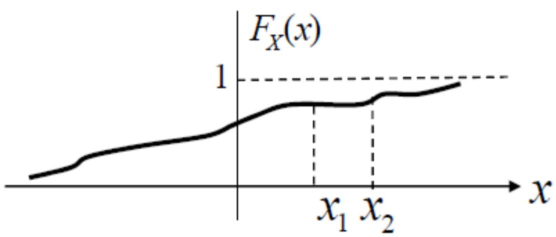
$$P\{X \leq x\} = F_X(x) \geq 0.$$

Where **x** is any real number

We call this function, denoted $F_X(x)$ the **cumulative probability distribution function (CDF)** of the random variable **X** (or just the **distribution function** of X)

- ✓ **Properties Distribution Functions:**

1. $F_X(-\infty) = P\{X \leq -\infty\} = P(\phi) = 0$
2. $F_X(\infty) = P\{X \leq \infty\} = 1$
3. $0 \leq F_X(x) \leq 1$
4. $F_X(x_1) \leq F_X(x_2) \quad \text{if} \quad x_1 < x_2$
5. $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
6. $F_X(x^+) \leq F_X(x)$



✓ Distribution Function for Discrete Random Variable:

- The distribution function of a **discrete random variable X** can be obtained from its **probability function** by noting that, for all **x** in $(-\infty\,,\ \infty)$

$$F(x) = P(X \le x) = \sum_{u \le x} f(u)$$

- If X takes on only a finite number of values $x_1, x_2, \ldots, x_n$, then the distribution function is given by

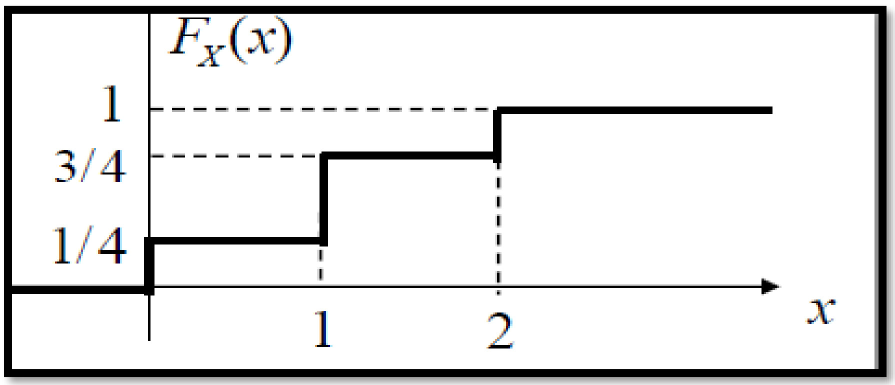
$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \le x < x_2 \\ f(x_1) + f(x_2) & x_2 \le x < x_3 \\ \vdots & \vdots \\ f(x_1) + \cdots + f(x_n) & x_n \le x < \infty \end{cases}$$

- The distribution function of a **discrete random variable X** is given by:

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\} u(x - x_i)$$

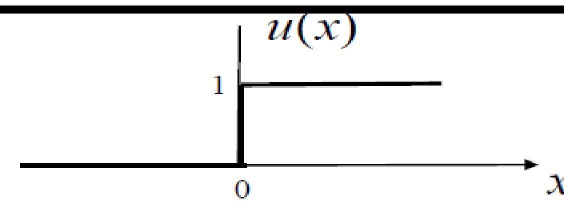
- **Example**:

$$F_X(x) = \frac{1}{4}u(x) + \frac{2}{4}u(x-1) + \frac{1}{4}u(x-2)$$



Where u(.) is the unit step defined by:

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



- **Example**: Suppose that a coin is tossed twice so that the sample space is **$S = \{HH, HT, TH, TT\}$**. Let **X** represent the number of heads that can come up.
  - It follows that X is a **random variable** as in the table

Sample Point	HH	HT	TH	TT
X	2	1	1	0

  - the **probability function** corresponding to the random variable **X**

$$P(HH) = \frac{1}{4} \qquad P(HT) = \frac{1}{4} \qquad P(TH) = \frac{1}{4} \qquad P(TT) = \frac{1}{4}$$

Then

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

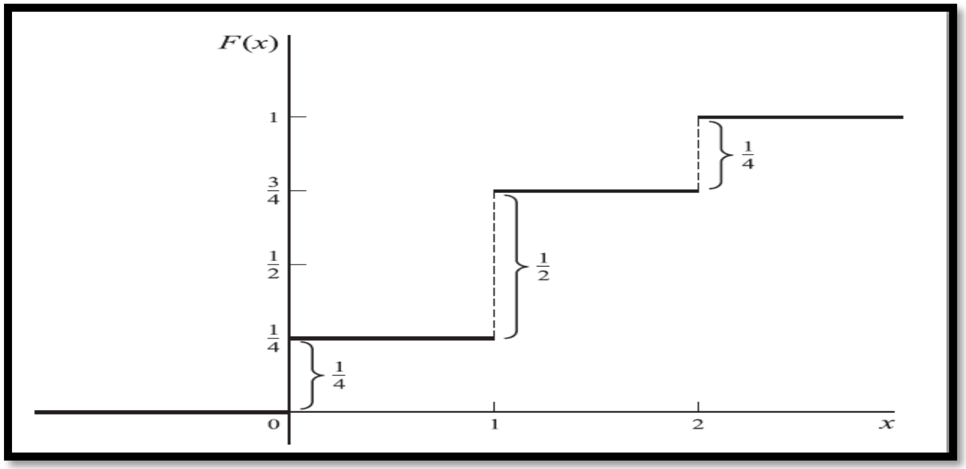
$$P(X = 2) = P(HH) = \frac{1}{4}$$

The probability function is thus given by

x	0	1	2
$f(x)$	1/4	1/2	1/4

- the distribution function for the random variable **X**

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & 2 \le x < \infty \end{cases}$$



Probability Density Function (PDF)

- ✓ Continuous Random Variables: A non-discrete random variable **X** is said to be absolutely continuous, or simply continuous, if its distribution function may be represented as

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du \qquad (-\infty < x < \infty)$$

where the function **f (x)** has the properties

$$1.\ f(x) \ge 0$$

$$2.\ \int_{-\infty}^{\infty} f(x)\,dx = 1$$

- ✓ A function **f (x)** is more often called a **probability density function** or simply density function
- ✓ Interval probability that X lies between two different values, say, a and b, is given by

$$P(a < X < b) = \int_{a}^{b} f(x)\,dx$$

✓The **probability density function** (or density function) (**PDF**) is defined as the derivative of the distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

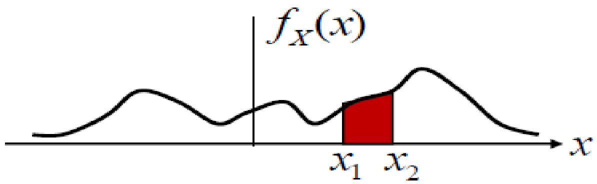
✓**Properties of Density Functions:**

1. $f_X(x) \geq 0 \quad \text{all } x$

2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

3. $F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$

4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)dx$



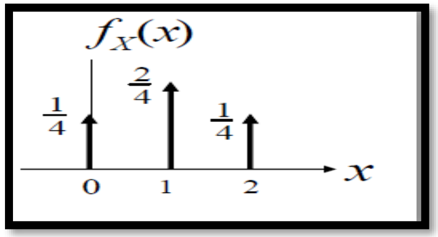
✓**Density Function for Discrete Random Variable (mass function):**

✓The density function of a discrete random variable X is given by:

$$f_X(x) = \sum_{i=1}^{N} P\{X = x_i\}\delta(x - x_i)$$

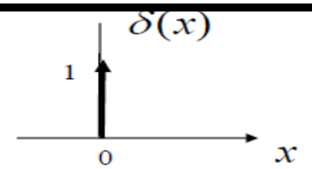
○ **Example**:

$$f_X(x) = \frac{1}{4}\delta(x) + \frac{2}{4}\delta(x-1) + \frac{1}{4}\delta(x-2)$$



Where $\delta(.)$ is the unit impulse defined by:

$$\delta(x) = \frac{du(x)}{dx}$$



○ **Example:** Let $\mathbf{X}$ have the discrete values in the set

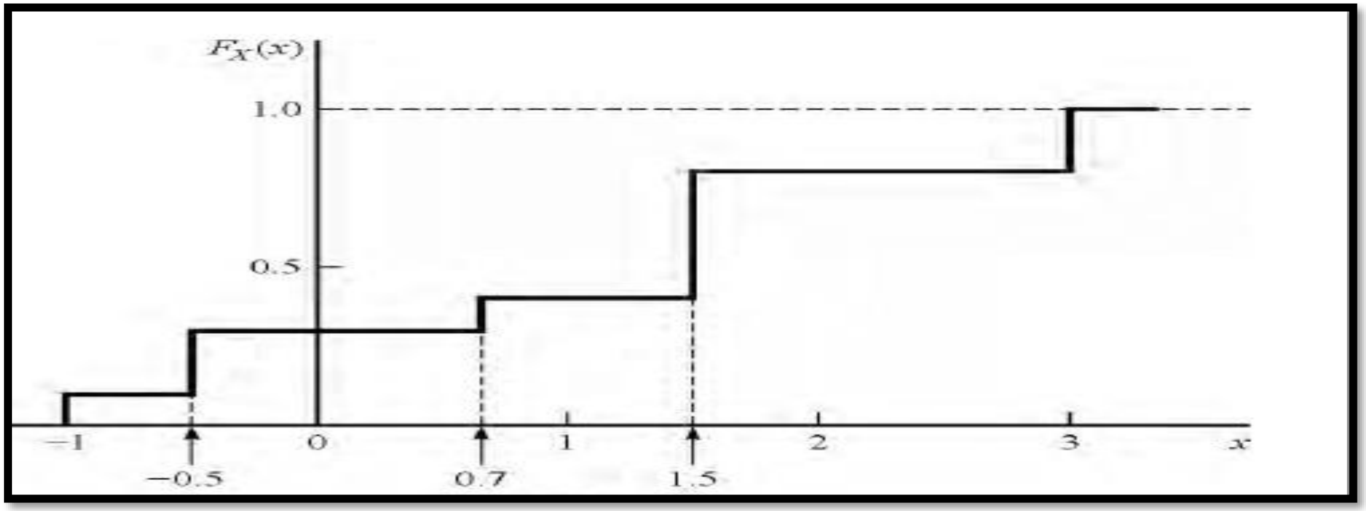
$\{-1, -0.5, 0.7, 1.5, 3\}$.

The corresponding probabilities are

$\{0.1, 0.2, 0.1, 0.4, 0.2\}$;

The distribution function:

$$F_X(x) = 0.1u(x+1) + 0.2u(x+0.5) + 0.1u(x-0.7) + 0.4u(x-1.5) + 0.2u(x-3)$$

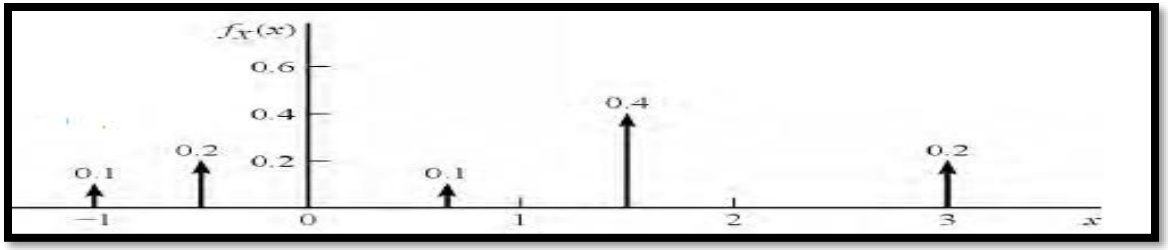


$$P(X \le 1.5) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$$

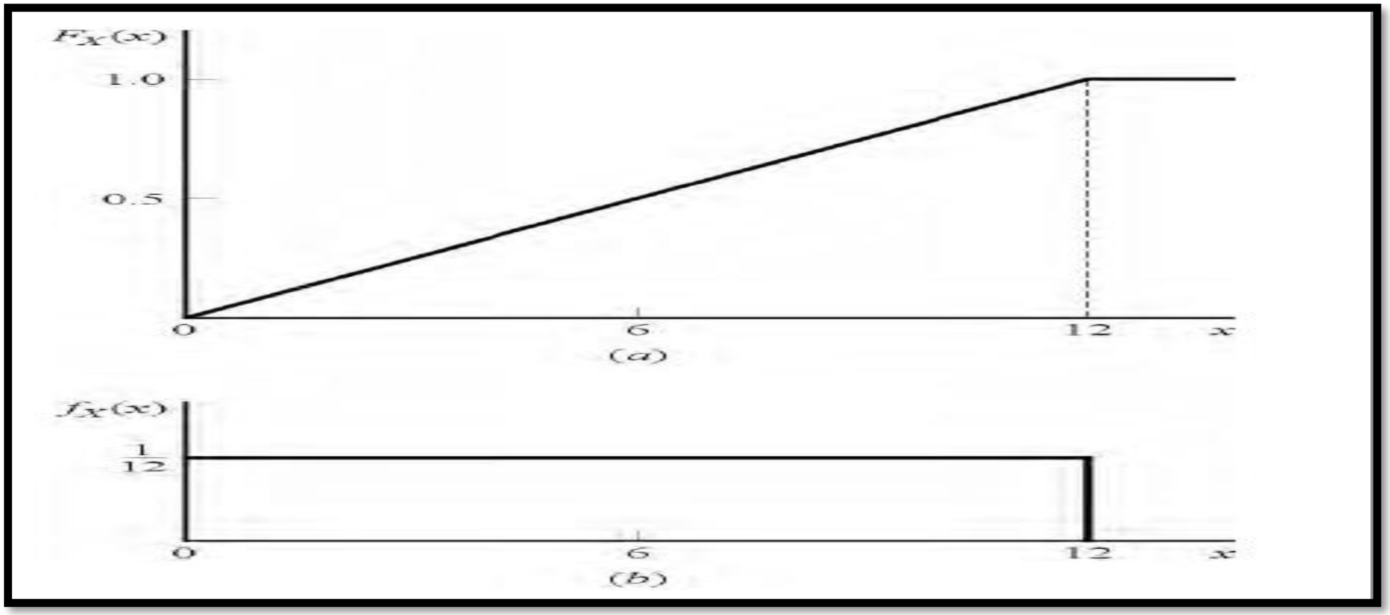
$$P(X \le 2) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$$

The density function:

$$f_X(x) = 0.1\delta(x+1) + 0.2\delta(x+0.5) + 0.1\delta(x-0.7) + 0.4\delta(x-1.5) + 0.2\delta(x-3)$$



○ **Example:** The corresponding distribution and the density functions for the wheel of chance experiment are shown in Fig.



$$P(X \le 6) = F_X(6) = 0.5$$

$$\text{OR} \quad = \int_{-\infty}^{6} f_X(x)\,dx = 0.5$$

○ **Example** :

a) Find the constant **c** such that the function

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Is a density function,

b) Compute $P(1 < X < 2)$.

c) Find the distribution function for the random variable

d) Use the result of (c) to find $P(1 < x \leq 2)$.

Solution:

a) using the 2 property

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{3} cx^2\,dx = \left.\frac{cx^3}{3}\right|_{0}^{3} = 9c$$

and since this must equal 1, we have **c = 1/9** b)

$$P(1 < X < 2) = \int_1^2 \frac{1}{9} x^2 \, dx = \frac{x^3}{27}\bigg|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$

c)

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\, du$$

If $x < 0$, then $F(x) = 0$.

If $0 \le x < 3$, then

$$F(x) = \int_0^x f(u)\, du = \int_0^x \frac{1}{9} u^2 \, du = \frac{x^3}{27}$$

If $x \ge 3$, then

$$F(x) = \int_0^3 f(u)\, du + \int_3^x f(u)\, du = \int_0^3 \frac{1}{9} u^2 \, du + \int_3^x 0 \, du = 1$$

Thus the required distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3/27 & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

d)

$$\begin{aligned} P(1 < X \le 2) &= P(X \le 2) - P(X \le 1) \\ &= F(2) - F(1) \\ &= \frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27} \end{aligned}$$